5.2 The Natural Logarithmic Function: Integration

- **Use the Log Rule for Integration to integrate a rational function.**
- **Integrate trigonometric functions.**

Exploration

Integrating Rational Functions
Early in Chapter 4, you learned rules that allowed you to integrate *any* polynomial function. The Log Rule presented in this section goes a long way toward enabling you to integrate rational functions. For instance, each of the following functions can be integrated with the Log Rule.

$\frac{2}{x}$	Example 1
$\frac{1}{4x-1}$	Example 2
$\frac{x}{x^2+1}$	Example 3
$\frac{3x^2+1}{x^3+x}$	Example 4(a)
$\frac{x+1}{x^2+2x}$	Example 4(c)
$\frac{1}{3x+2}$	Example 4(d)
$\frac{x^2+x+1}{x^2+1}$	Example 5
$\frac{2x}{(x+1)^2}$	Example 6

There are still some rational functions that cannot be integrated using the Log Rule. Give examples of these functions, and explain your reasoning.

Log Rule for Integration

The differentiation rules

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

that you studied in the preceding section produce the following integration rule.

THEOREM 5.5 Log Rule for Integration

Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,dx = \ln|x| + C$ **2.** $\int \frac{1}{u}\,du = \ln|u| + C$

Because $du = u'\,dx$, the second formula can also be written as

$$\int \frac{u'}{u}\,dx = \ln|u| + C. \qquad \text{Alternative form of Log Rule}$$

EXAMPLE 1 Using the Log Rule for Integration

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx \qquad \text{Constant Multiple Rule}$$
$$= 2\ln|x| + C \qquad \text{Log Rule for Integration}$$
$$= \ln(x^2) + C \qquad \text{Property of logarithms}$$

Because x^2 cannot be negative, the absolute value notation is unnecessary in the final form of the antiderivative.

EXAMPLE 2 Using the Log Rule with a Change of Variables

Find $\int \frac{1}{4x-1}\,dx$.

Solution If you let $u = 4x - 1$, then $du = 4\,dx$.

$$\int \frac{1}{4x-1}\,dx = \frac{1}{4}\int\left(\frac{1}{4x-1}\right)4\,dx \qquad \text{Multiply and divide by 4.}$$
$$= \frac{1}{4}\int \frac{1}{u}\,du \qquad \text{Substitute: } u = 4x - 1.$$
$$= \frac{1}{4}\ln|u| + C \qquad \text{Apply Log Rule.}$$
$$= \frac{1}{4}\ln|4x-1| + C \qquad \text{Back-substitute.}$$

Example 3 uses the alternative form of the Log Rule. To apply this rule, look for quotients in which the numerator is the derivative of the denominator.

EXAMPLE 3 Finding Area with the Log Rule

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2 + 1}$$

the x-axis, and the line $x = 3$.

Solution In Figure 5.8, you can see that the area of the region is given by the definite integral

$$\int_0^3 \frac{x}{x^2 + 1}\,dx.$$

If you let $u = x^2 + 1$, then $u' = 2x$. To apply the Log Rule, multiply and divide by 2 as shown.

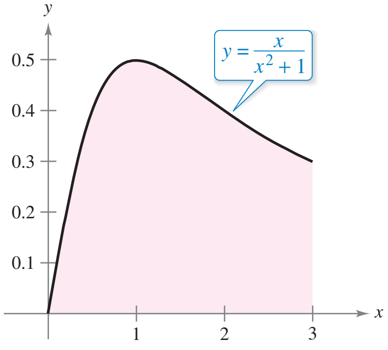
$$\int_0^3 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^3 \frac{2x}{x^2+1}\,dx \qquad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2}\Big[\ln(x^2+1)\Big]_0^3 \qquad \int \frac{u'}{u}\,dx = \ln|u| + C$$

$$= \frac{1}{2}(\ln 10 - \ln 1)$$

$$= \frac{1}{2}\ln 10 \qquad \ln 1 = 0$$

$$\approx 1.151$$



Area $= \int_0^3 \frac{x}{x^2+1}\,dx$

The area of the region bounded by the graph of y, the x-axis, and $x = 3$ is $\frac{1}{2}\ln 10$.

Figure 5.8

EXAMPLE 4 Recognizing Quotient Forms of the Log Rule

a. $\int \frac{3x^2 + 1}{x^3 + x}\,dx = \ln|x^3 + x| + C \qquad u = x^3 + x$

b. $\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + C \qquad u = \tan x$

c. $\int \frac{x + 1}{x^2 + 2x}\,dx = \frac{1}{2}\int \frac{2x + 2}{x^2 + 2x}\,dx \qquad u = x^2 + 2x$

$$= \frac{1}{2}\ln|x^2 + 2x| + C$$

d. $\int \frac{1}{3x + 2}\,dx = \frac{1}{3}\int \frac{3}{3x + 2}\,dx \qquad u = 3x + 2$

$$= \frac{1}{3}\ln|3x + 2| + C$$

With antiderivatives involving logarithms, it is easy to obtain forms that look quite different but are still equivalent. For instance, both

$$\ln|(3x + 2)^{1/3}| + C$$

and

$$\ln|3x + 2|^{1/3} + C$$

are equivalent to the antiderivative listed in Example 4(d).

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, when a rational function has a *numerator of degree greater than or equal to that of the denominator*, division may reveal a form to which you can apply the Log Rule. This is shown in Example 5.

EXAMPLE 5 Using Long Division Before Integrating

See LarsonCalculus.com for an interactive version of this type of example.

Find the indefinite integral.

$$\int \frac{x^2 + x + 1}{x^2 + 1}\,dx$$

Solution Begin by using long division to rewrite the integrand.

$$\frac{x^2 + x + 1}{x^2 + 1} \quad \Longrightarrow \quad \begin{array}{r} 1 \\ x^2 + 1 \,\big)\, \overline{x^2 + x + 1} \\ \underline{x^2 \qquad + 1} \\ x \end{array} \quad \Longrightarrow \quad 1 + \frac{x}{x^2 + 1}$$

Now, you can integrate to obtain

$$\int \frac{x^2 + x + 1}{x^2 + 1}\,dx = \int \left(1 + \frac{x}{x^2 + 1}\right) dx \qquad \text{Rewrite using long division.}$$

$$= \int dx + \frac{1}{2}\int \frac{2x}{x^2 + 1}\,dx \qquad \text{Rewrite as two integrals.}$$

$$= x + \frac{1}{2}\ln(x^2 + 1) + C. \qquad \text{Integrate.}$$

Check this result by differentiating to obtain the original integrand. ■

The next example presents another instance in which the use of the Log Rule is disguised. In this case, a change of variables helps you recognize the Log Rule.

EXAMPLE 6 Change of Variables with the Log Rule

Find the indefinite integral.

$$\int \frac{2x}{(x + 1)^2}\,dx$$

Solution If you let $u = x + 1$, then $du = dx$ and $x = u - 1$.

$$\int \frac{2x}{(x + 1)^2}\,dx = \int \frac{2(u - 1)}{u^2}\,du \qquad \text{Substitute.}$$

$$= 2\int \left(\frac{u}{u^2} - \frac{1}{u^2}\right) du \qquad \text{Rewrite as two fractions.}$$

$$= 2\int \frac{du}{u} - 2\int u^{-2}\,du \qquad \text{Rewrite as two integrals.}$$

$$= 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C \qquad \text{Integrate.}$$

$$= 2\ln|u| + \frac{2}{u} + C \qquad \text{Simplify.}$$

$$= 2\ln|x + 1| + \frac{2}{x + 1} + C \qquad \text{Back-substitute.}$$

Check this result by differentiating to obtain the original integrand. ■

TECHNOLOGY If you have access to a computer algebra system, use it to find the indefinite integrals in Examples 5 and 6. How does the form of the antiderivative that it gives you compare with that given in Examples 5 and 6?

As you study the methods shown in Examples 5 and 6, be aware that both methods involve rewriting a disguised integrand so that it fits one or more of the basic integration formulas. Throughout the remaining sections of Chapter 5 and in Chapter 8, much time will be devoted to integration techniques. To master these techniques, you must recognize the "form-fitting" nature of integration. In this sense, integration is not nearly as straightforward as differentiation. Differentiation takes the form

"Here is the question; what is the answer?"

Integration is more like

"Here is the answer; what is the question?"

Here are some guidelines you can use for integration.

GUIDELINES FOR INTEGRATION

1. Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and 10 trigonometric rules. By the end of Section 5.7, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. When you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

EXAMPLE 7 ***u*-Substitution and the Log Rule**

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}$.

Solution The solution can be written as an indefinite integral.

$$y = \int \frac{1}{x \ln x}\, dx$$

Because the integrand is a quotient whose denominator is raised to the first power, you should try the Log Rule. There are three basic choices for u. The choices

$$u = x \quad \text{and} \quad u = x \ln x$$

fail to fit the u'/u form of the Log Rule. However, the third choice does fit. Letting $u = \ln x$ produces $u' = 1/x$, and you obtain the following.

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1/x}{\ln x}\, dx \qquad \text{Divide numerator and denominator by } x.$$

$$= \int \frac{u'}{u}\, dx \qquad \text{Substitute: } u = \ln x.$$

$$= \ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= \ln|\ln x| + C \qquad \text{Back-substitute.}$$

So, the solution is $y = \ln|\ln x| + C$.

REMARK Keep in mind that you can check your answer to an integration problem by differentiating the answer. For instance, in Example 7, the derivative of $y = \ln|\ln x| + C$ is $y' = 1/(x \ln x)$.

Integrals of Trigonometric Functions

In Section 4.1, you looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, you can now complete the set of basic trigonometric integration formulas.

EXAMPLE 8 Using a Trigonometric Identity

Find $\int \tan x \, dx$.

Solution This integral does not seem to fit any formulas on our basic list. However, by using a trigonometric identity, you obtain

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.$$

Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx$$ Apply trigonometric identity and multiply and divide by -1.

$$= -\int \frac{u'}{u} \, dx$$ Substitute: $u = \cos x$.

$$= -\ln|u| + C$$ Apply Log Rule.

$$= -\ln|\cos x| + C.$$ Back-substitute.

Example 8 uses a trigonometric identity to derive an integration rule for the tangent function. The next example takes a rather unusual step (multiplying and dividing by the same quantity) to derive an integration rule for the secant function.

EXAMPLE 9 Derivation of the Secant Formula

Find $\int \sec x \, dx$.

Solution Consider the following procedure.

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

Letting u be the denominator of this quotient produces

$$u = \sec x + \tan x$$

and

$$u' = \sec x \tan x + \sec^2 x.$$

So, you can conclude that

$$\int \sec x \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$ Rewrite integrand.

$$= \int \frac{u'}{u} \, dx$$ Substitute: $u = \sec x + \tan x$.

$$= \ln|u| + C$$ Apply Log Rule.

$$= \ln|\sec x + \tan x| + C.$$ Back-substitute.

With the results of Examples 8 and 9, you now have integration formulas for $\sin x$, $\cos x$, $\tan x$, and $\sec x$. The integrals of the six basic trigonometric functions are summarized below. (For proofs of $\cot u$ and $\csc u$, see Exercises 87 and 88.)

REMARK Using trigonometric identities and properties of logarithms, you could rewrite these six integration rules in other forms. For instance, you could write

$$\int \csc u\, du = \ln|\csc u - \cot u| + C.$$

(See Exercises 89–92.)

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u\, du = -\cos u + C \qquad \int \cos u\, du = \sin u + C$$

$$\int \tan u\, du = -\ln|\cos u| + C \qquad \int \cot u\, du = \ln|\sin u| + C$$

$$\int \sec u\, du = \ln|\sec u + \tan u| + C \qquad \int \csc u\, du = -\ln|\csc u + \cot u| + C$$

EXAMPLE 10 Integrating Trigonometric Functions

Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$.

Solution Using $1 + \tan^2 x = \sec^2 x$, you can write

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx &= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx \\
&= \int_0^{\pi/4} \sec x\, dx && \sec x \geq 0 \text{ for } 0 \leq x \leq \frac{\pi}{4}. \\
&= \ln|\sec x + \tan x| \Big]_0^{\pi/4} \\
&= \ln\left(\sqrt{2} + 1\right) - \ln 1 \\
&\approx 0.881.
\end{aligned}$$

EXAMPLE 11 Finding an Average Value

Find the average value of

$$f(x) = \tan x$$

on the interval $[0, \pi/4]$.

Solution

$$\begin{aligned}
\text{Average value} &= \frac{1}{(\pi/4) - 0} \int_0^{\pi/4} \tan x\, dx && \text{Average value} = \frac{1}{b-a}\int_a^b f(x)\, dx \\
&= \frac{4}{\pi}\int_0^{\pi/4} \tan x\, dx && \text{Simplify.} \\
&= \frac{4}{\pi}\Big[-\ln|\cos x|\Big]_0^{\pi/4} && \text{Integrate.} \\
&= -\frac{4}{\pi}\left[\ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1)\right] \\
&= -\frac{4}{\pi}\ln\left(\frac{\sqrt{2}}{2}\right) \\
&\approx 0.441
\end{aligned}$$

The average value is about 0.441, as shown in Figure 5.9.

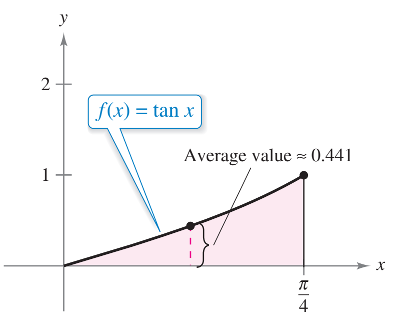


Figure 5.9

5.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding an Indefinite Integral In Exercises 1–26, find the indefinite integral.

1. $\displaystyle\int \frac{5}{x}\,dx$
2. $\displaystyle\int \frac{10}{x}\,dx$
3. $\displaystyle\int \frac{1}{x+1}\,dx$
4. $\displaystyle\int \frac{1}{x-5}\,dx$
5. $\displaystyle\int \frac{1}{2x+5}\,dx$
6. $\displaystyle\int \frac{9}{5-4x}\,dx$
7. $\displaystyle\int \frac{x}{x^2-3}\,dx$
8. $\displaystyle\int \frac{x^2}{5-x^3}\,dx$
9. $\displaystyle\int \frac{4x^3+3}{x^4+3x}\,dx$
10. $\displaystyle\int \frac{x^2-2x}{x^3-3x^2}\,dx$
11. $\displaystyle\int \frac{x^2-4}{x}\,dx$
12. $\displaystyle\int \frac{x^3-8x}{x^2}\,dx$
13. $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx$
14. $\displaystyle\int \frac{x^2+4x}{x^3+6x^2+5}\,dx$
15. $\displaystyle\int \frac{x^2-3x+2}{x+1}\,dx$
16. $\displaystyle\int \frac{2x^2+7x-3}{x-2}\,dx$
17. $\displaystyle\int \frac{x^3-3x^2+5}{x-3}\,dx$
18. $\displaystyle\int \frac{x^3-6x-20}{x+5}\,dx$
19. $\displaystyle\int \frac{x^4+x-4}{x^2+2}\,dx$
20. $\displaystyle\int \frac{x^3-4x^2-4x+20}{x^2-5}\,dx$
21. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$
22. $\displaystyle\int \frac{1}{x\ln x^3}\,dx$
23. $\displaystyle\int \frac{1}{\sqrt{x}(1-3\sqrt{x})}\,dx$
24. $\displaystyle\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx$
25. $\displaystyle\int \frac{2x}{(x-1)^2}\,dx$
26. $\displaystyle\int \frac{x(x-2)}{(x-1)^3}\,dx$

Finding an Indefinite Integral by *u*-Substitution In Exercises 27–30, find the indefinite integral by *u*-substitution. (*Hint:* Let *u* be the denominator of the integrand.)

27. $\displaystyle\int \frac{1}{1+\sqrt{2x}}\,dx$
28. $\displaystyle\int \frac{1}{1+\sqrt{3x}}\,dx$
29. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx$
30. $\displaystyle\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx$

Finding an Indefinite Integral of a Trigonometric Function In Exercises 31–40, find the indefinite integral.

31. $\displaystyle\int \cot\frac{\theta}{3}\,d\theta$
32. $\displaystyle\int \tan 5\theta\,d\theta$
33. $\displaystyle\int \csc 2x\,dx$
34. $\displaystyle\int \sec\frac{x}{2}\,dx$
35. $\displaystyle\int (\cos 3\theta - 1)\,d\theta$
36. $\displaystyle\int \left(2-\tan\frac{\theta}{4}\right)d\theta$
37. $\displaystyle\int \frac{\cos t}{1+\sin t}\,dt$
38. $\displaystyle\int \frac{\csc^2 t}{\cot t}\,dt$
39. $\displaystyle\int \frac{\sec x\tan x}{\sec x - 1}\,dx$
40. $\displaystyle\int (\sec 2x + \tan 2x)\,dx$

Differential Equation In Exercises 41–44, solve the differential equation. Use a graphing utility to graph three solutions, one of which passes through the given point.

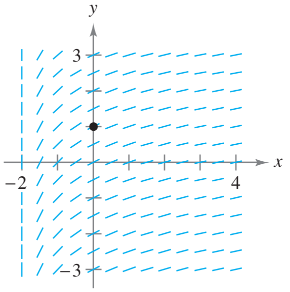
41. $\displaystyle\frac{dy}{dx} = \frac{3}{2-x},\quad (1, 0)$
42. $\displaystyle\frac{dy}{dx} = \frac{x-2}{x},\quad (-1, 0)$
43. $\displaystyle\frac{dy}{dx} = \frac{2x}{x^2-9x},\quad (0, 4)$
44. $\displaystyle\frac{dr}{dt} = \frac{\sec^2 t}{\tan t + 1},\quad (\pi, 4)$

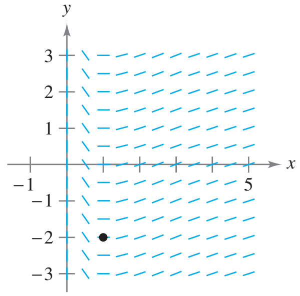
Finding a Particular Solution In Exercises 45 and 46, find the particular solution that satisfies the differential equation and the initial equations.

45. $f''(x) = \dfrac{2}{x^2},\ f'(1) = 1,\ f(1) = 1,\ x > 0$
46. $f''(x) = -\dfrac{4}{(x-1)^2} - 2,\ f'(2) = 0,\ f(2) = 3,\ x > 1$

Slope Field In Exercises 47 and 48, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to *MathGraphs.com.*

47. $\displaystyle\frac{dy}{dx} = \frac{1}{x+2},\quad (0, 1)$
48. $\displaystyle\frac{dy}{dx} = \frac{\ln x}{x},\quad (1, -2)$





Evaluating a Definite Integral In Exercises 49–56, evaluate the definite integral. Use a graphing utility to verify your result.

49. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx$
50. $\displaystyle\int_{-1}^1 \frac{1}{2x+3}\,dx$

51. $\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx$

52. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

53. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx$

54. $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx$

55. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta$

56. $\displaystyle\int_{\pi/8}^{\pi/4} (\csc 2\theta - \cot 2\theta)\,d\theta$

Using Technology to Find an Integral **In Exercises 57–62, use a computer algebra system to find or evaluate the integral.**

57. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

58. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

59. $\displaystyle\int \frac{\sqrt{x}}{x-1}\,dx$

60. $\displaystyle\int \frac{x^2}{x-1}\,dx$

61. $\displaystyle\int_{\pi/4}^{\pi/2} (\csc x - \sin x)\,dx$

62. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx$

Finding a Derivative **In Exercises 63–66, find $F'(x)$.**

63. $\displaystyle F(x) = \int_1^x \frac{1}{t}\,dt$

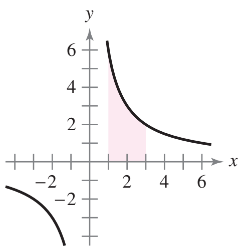
64. $\displaystyle F(x) = \int_0^x \tan t\,dt$

65. $\displaystyle F(x) = \int_1^{3x} \frac{1}{t}\,dt$

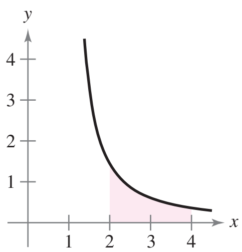
66. $\displaystyle F(x) = \int_1^{x^2} \frac{1}{t}\,dt$

Area **In Exercises 67–70, find the area of the given region. Use a graphing utility to verify your result.**

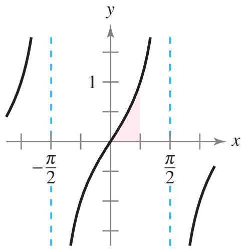
67. $y = \dfrac{6}{x}$



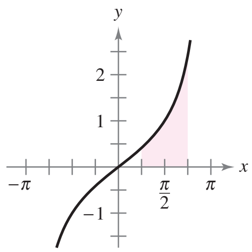
68. $y = \dfrac{2}{x\ln x}$



69. $y = \tan x$



70. $y = \dfrac{\sin x}{1+\cos x}$



Area **In Exercises 71–74, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.**

71. $y = \dfrac{x^2+4}{x}, \quad x = 1, \quad x = 4, \quad y = 0$

72. $y = \dfrac{5x}{x^2+2}, \quad x = 1, \quad x = 5, \quad y = 0$

73. $y = 2\sec\dfrac{\pi x}{6}, \quad x = 0, \quad x = 2, \quad y = 0$

74. $y = 2x - \tan 0.3x, \quad x = 1, \quad x = 4, \quad y = 0$

Numerical Integration **In Exercises 75–78, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral. Let $n = 4$ and round your answer to four decimal places. Use a graphing utility to verify your result.**

75. $\displaystyle\int_1^5 \frac{12}{x}\,dx$

76. $\displaystyle\int_0^4 \frac{8x}{x^2+4}\,dx$

77. $\displaystyle\int_2^6 \ln x\,dx$

78. $\displaystyle\int_{-\pi/3}^{\pi/3} \sec x\,dx$

WRITING ABOUT CONCEPTS

Choosing a Formula **In Exercises 79–82, state the integration formula you would use to perform the integration. Do not integrate.**

79. $\displaystyle\int \sqrt[3]{x}\,dx$

80. $\displaystyle\int \frac{x}{(x^2+4)^3}\,dx$

81. $\displaystyle\int \frac{x}{x^2+4}\,dx$

82. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$

Approximation **In Exercises 83 and 84, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region, not by performing any calculations.)**

83. $f(x) = \sec x, \quad [0, 1]$

(a) 6 (b) -6 (c) $\frac{1}{2}$ (d) 1.25 (e) 3

84. $f(x) = \dfrac{2x}{x^2+1}, \quad [0, 4]$

(a) 3 (b) 7 (c) -2 (d) 5 (e) 1

85. Finding a Value Find a value of x such that

$$\int_1^x \frac{3}{t}\,dt = \int_{1/4}^x \frac{1}{t}\,dt.$$

86. Finding a Value Find a value of x such that

$$\int_1^x \frac{1}{t}\,dt$$

is equal to (a) $\ln 5$ and (b) 1.

87. Proof Prove that

$$\int \cot u\,du = \ln|\sin u| + C.$$

88. Proof Prove that

$$\int \csc u\,du = -\ln|\csc u + \cot u| + C.$$

Using Properties of Logarithms and Trigonometric Identities **In Exercises 89–92, show that the two formulas are equivalent.**

89. $\int \tan x\,dx = -\ln|\cos x| + C$

$\int \tan x\,dx = \ln|\sec x| + C$

90. $\int \cot x\,dx = \ln|\sin x| + C$

$\int \cot x\,dx = -\ln|\csc x| + C$

91. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

$\int \sec x\,dx = -\ln|\sec x - \tan x| + C$

92. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

$\int \csc x\,dx = \ln|\csc x - \cot x| + C$

Finding the Average Value of a Function **In Exercises 93–96, find the average value of the function over the given interval.**

93. $f(x) = \dfrac{8}{x^2}, \quad [2, 4]$

94. $f(x) = \dfrac{4(x+1)}{x^2}, \quad [2, 4]$

95. $f(x) = \dfrac{2\ln x}{x}, \quad [1, e]$

96. $f(x) = \sec\dfrac{\pi x}{6}, \quad [0, 2]$

97. Population Growth A population of bacteria P is changing at a rate of

$$\frac{dP}{dt} = \frac{3000}{1 + 0.25t}$$

where t is the time in days. The initial population (when $t = 0$) is 1000. Write an equation that gives the population at any time t. Then find the population when $t = 3$ days.

98. Sales The rate of change in sales S is inversely proportional to time t $(t > 1)$, measured in weeks. Find S as a function of t when the sales after 2 and 4 weeks are 200 units and 300 units, respectively.

99. Heat Transfer

Find the time required for an object to cool from 300°F to 250°F by evaluating

$$t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T - 100}\,dT$$

where t is time in minutes.

100. Average Price The demand equation for a product is

$$p = \frac{90{,}000}{400 + 3x}$$

where p is the price (in dollars) and x is the number of units (in thousands). Find the average price p on the interval $40 \le x \le 50$.

101. Area and Slope Graph the function

$$f(x) = \frac{x}{1 + x^2}$$

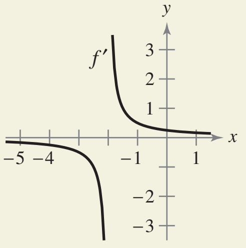
on the interval $[0, \infty)$.

(a) Find the area bounded by the graph of f and the line $y = \frac{1}{2}x$.

(b) Determine the values of the slope m such that the line $y = mx$ and the graph of f enclose a finite region.

(c) Calculate the area of this region as a function of m.

102. HOW DO YOU SEE IT? Use the graph of f' shown in the figure to answer the following.



(a) Approximate the slope of f at $x = -1$. Explain.

(b) Approximate any open intervals in which the graph of f is increasing and any open intervals in which it is decreasing. Explain.

True or False? **In Exercises 103–106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

103. $(\ln x)^{1/2} = \frac{1}{2}\ln x$

104. $\int \ln x\,dx = (1/x) + C$

105. $\int \frac{1}{x}\,dx = \ln|cx|, \quad c \ne 0$

106. $\int_{-1}^{2} \frac{1}{x}\,dx = \Big[\ln|x|\Big]_{-1}^{2} = \ln 2 - \ln 1 = \ln 2$

107. Napier's Inequality For $0 < x < y$, show that

$$\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}.$$

108. Proof Prove that the function

$$F(x) = \int_{x}^{2x} \frac{1}{t}\,dt$$

is constant on the interval $(0, \infty)$.

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